Online Appendix to:
Partial Identification of Spread Parameters

Jörg Stoye

Cornell University

June 3, 2013

Closed-form Bounds on Some Spread Parameters with Missing Data

This appendix collects some closed-form characterizations of bounds that are omitted from the paper, although some are implemented in the empirical example. The setting is the one of section 4, thus there is no covariate, $Y \in [0,1]$, and the observable data have distribution F_1 and probability mass p. Assume first Dominant Selection, then substituting from the display on page 17 into theorems 2 and 3 leads to the following explicit (and in some cases, closed-form) characterizations.

Proposition 1 ***Constrained Bounds on Spread Parameters with Dominant Selection***

Assume that supp$(Y) = [0,1]$*, that* F_1 *and* p *are known, and that Dominant Selection applies.*

(i) *Let* $E(Y) = \mu$ *for some pre-assigned* $\mu \in H(E(Y)) = [pE_1(Y), E_1(Y)]$*, and let* θ *be any* D_2*-parameter. Define* $\underline{F}_\mu$ *and* $\overline{F}_\mu$ *as follows:*

$$\underline{F}_\mu(y) = \begin{cases} F(y), & y < Q(\alpha) \\ 1, & y \geq Q(\alpha) \end{cases}$$

$$\overline{F}_\mu(y) = \begin{cases} \beta, & y < Q(\beta) \\ F(y), & y \geq Q(\beta) \end{cases},$$

where α *and* β *are implicitly defined by* $\underline{E}_\mu(Y) = \overline{E}_\mu(Y) = (\mu - pE_1(Y))/(1-p)$*. Then*

$$\theta\left(pF_1 + (1-p)\underline{F}_\mu\right) \leq \theta(F) \leq \theta\left(pF_1 + (1-p)\overline{F}_\mu\right).$$

(ii) *Let* $\theta = f(Q(\alpha), Q(\beta))$ *be a quantile contrast, let* F_1 *be continuous with full support, and let* $Q(\gamma) = m$ *for some pre-assigned* $\gamma \in (\alpha, \beta)$ *and* $m \in H(Q(\gamma)) = [Q_1(1-(1-\gamma)/p), Q_1(\gamma)]$*. Then:*

$$f(\min\{Q_1(\alpha), m\}, \max\{Q_1(1-(1-\beta)/p), m\}) \leq \theta \leq f(Q_1(F_1(m)-(\gamma-\alpha)/p), Q_1(\min\{\beta, F_1(m)+(\beta-\gamma)/p\})).$$

More closed-form analysis is possible under Limited Selection. Thus, now assume that LS(k) holds. Using lemma 6 and theorems 2 and 3, one can then show the following.

Proposition 2 ***Bounds on Probabilities***

Let $P_1(A)$ and p be known and let LS(k) hold. Then

$$\max\left\{1-k(1-P_1(A)), pP_1(A)\right\} \leq \Pr(Y \in A) \leq \min\left\{kP_1(A), p\Pr P_1(A)+1-p\right\}.$$

Proposition 3 ***Bounds on*** $\mathbf{D_1}$***-Parameters***

Let θ be a D_1-parameter, let P_1 and p be known and let LS(k) holds. Then

$$\theta(\underline{F}) \leq \theta(F) \leq \theta(\overline{F}),$$

where $\underline{F}$ and $\overline{F}$ are characterized as follows:

$$\begin{aligned} \underline{F}(y) &= \min\left\{kF_1(y), pF_1(y)+1-p\right\} \\ \overline{F}(y) &= \max\left\{pF_1(y), 1-k(1-F_1(y))\right\}. \end{aligned}$$

These bounds imply that

$$Q_1\left(\max\left\{\frac{\alpha}{k}, 1-\frac{1-\alpha}{p}\right\}\right) \leq Q(\alpha) \leq Q_1\left(\min\left\{\frac{\alpha}{p}, 1-\frac{1-\alpha}{k}\right\}\right)$$

and if F_1 is continuous, they also imply

$$pE_1(Y)+(1-p)E_1\left(Y|Y \leq Q_1\left(\frac{1-p}{k-p}\right)\right) \leq E(Y) \leq pE_1(Y)+(1-p)E_1\left(Y|Y \geq Q_1\left(\frac{k-1}{k-p}\right)\right).$$

Proposition 4 Constrained Bounds on Spread Parameters

Let F_1 and p be known and let LS(k) hold.

(i) *Let θ be a D_2-parameter and let $E(Y)=\mu$ for some pre-assigned $\mu \in H(E(Y))$. Then*

$$\theta(\underline{F}_\mu) \leq \theta(F) \leq \theta(\overline{F}_\mu),$$

where $\underline{F}_\mu$ and $\overline{F}_\mu$ are characterized as follows:

$$\begin{aligned} \underline{F}_\mu(y) &= \begin{cases} pF_1(y), & y < Q_1(\underline{\alpha}) \\ pF_1(y)+\min\left\{(k-p)\left(F_1(y)-\underline{\alpha}\right), 1-p\right\}, & Q_1(\underline{\alpha}) \leq y \end{cases} \\ \overline{F}_\mu(y) &= \begin{cases} pF_1(y)+(k-p)\min\left\{F_1(y), \overline{\alpha}\right\}, & y < Q_1\left(\overline{\alpha}+\frac{k-1}{k-p}\right) \\ 1-k(1-F_1(y)), & y \geq Q_1\left(\overline{\alpha}+\frac{k-1}{k-p}\right) \end{cases}, \end{aligned}$$

and $\underline{\alpha} \in \left[0, \frac{k-1}{k-p}\right]$ and $\overline{\alpha} \in \left[0, \frac{1-p}{k-p}\right]$ are implicitly defined by $\underline{E}_\mu(Y) = \overline{E}_\mu(Y) = \mu$.

(ii) *Let* $\theta = f(Q(\alpha), Q(\beta))$ *be a quantile contrast, let* P_1 *be continuous with full support, and let* $Q(\gamma) = m$ *for some pre-assigned* $\gamma \in (\alpha, \beta)$ *and* $m \in H(Q(\gamma))$*. Then*

$$f\left(Q_1\left(\min\left\{F_1(m) - \frac{\gamma-\alpha}{k}, \frac{\alpha}{p}\right\}\right), Q_1\left(\max\left\{F_1(m) + \frac{\beta-\gamma}{k}, 1 - \frac{1-\beta}{p}\right\}\right)\right)$$

$$\leq \theta \leq$$

$$f\left(Q_1\left(\max\left\{F_1(m) - \frac{\gamma-\alpha}{p}, \frac{\alpha}{k}\right\}\right), Q_1\left(\min\left\{F_1(m) + \frac{\beta-\gamma}{p}, 1 - \frac{1-\beta}{k}\right\}\right)\right).$$